Part 2.3 Monotonic functions and their properties

v1 2019-20

Monotonic functions

Definition 2.3.1 *A function is*

- ***increasing*** *if* $x_1 < x_2$ *implies* $f(x_1) \leq f(x_2)$,
- ***strictly increasing*** *if* $x_1 < x_2$ *implies* $f(x_1) < f(x_2)$,
- ***decreasing*** *if* $x_1 < x_2$ *implies* $f(x_1) \geq f(x_2)$,
- ***strictly decreasing*** *if* $x_1 < x_2$ *implies* $f(x_1) > f(x_2)$,
- ***monotonic*** *if it is either increasing or decreasing,*
- ***strictly monotonic*** *if it is either strictly increasing or strictly decreasing,*

Example 2.3.2 *Verify the following.*

1. e^x *is strictly increasing on* $\mathbb{R}$, *(you may assume again that* $e^x e^y = e^{x+y}$ *for all* $x, y \in \mathbb{R}$ *and* $e^x > 1$ for $x > 0$.*)*
2. x^3 *is strictly increasing on* $\mathbb{R}$,
3. x^2 *is not increasing on* $\mathbb{R}$,
4. *for all* $n \in \mathbb{N}, x^n$ *is strictly increasing on* $[0, \infty)$.

Solutions

1. Assume $x < y$ then $e^y = e^{y-x+x} = e^{y-x}e^x > e^x$ by given assumptions, noting that $y - x > 0$.

2. Assume $x < y$ then consider

$$y^3 - x^3 = (y - x)\left(y^2 + yx + x^2\right).$$

The first factor is > 0 since $x > y$ but is the second factor > 0? If y and x are of different signs perhaps the term xy will be so large and negative that is dominates $y^2 + x^2$ making the factor negative. This does NOT happen.

The easiest way to show this (though it is a bit of a trick) is to complete the square

$$y^2 + yx + x^2 = \left(y + \frac{x}{2}\right)^2 + \frac{3x^2}{4}.$$

This is ≥ 0 since squares are non-negative but it is, in fact, > 0 since we can have this equal to zero iff $y = x = 0$ and we have demanded that $y > x$.

3. Give a counterexample, i.e. $-2 < -1$ yet $(-2)^2 > (-1)^2$.

4. On Question Sheet. ■

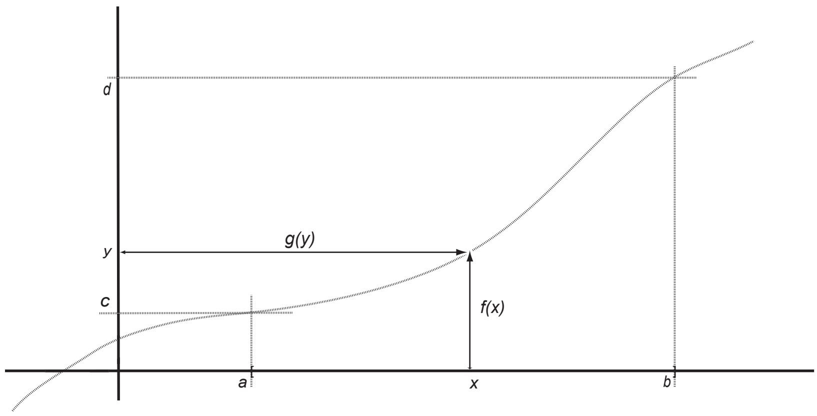
Theorem 2.3.3 ***Inverse Function Theorem***

Assume that f is continuous and strictly monotonic on the closed and bounded interval $[a, b]$. Write

$$[c, d] = \begin{cases} [f(a), f(b)] & \text{if } f \text{ is increasing} \\ [f(b), f(a)] & \text{if } f \text{ is decreasing.} \end{cases}$$

Then there exists a function g, continuous and strictly monotonic on $[c, d]$ which is inverse to f, i.e. $g(f(x)) = x$ for all $x \in [a, b]$ and $f(g(y)) = y$ for all $y \in [c, d]$.

Proof *Not given in 2018-19* Assume that f is strictly increasing.



Existence of the inverse.

• $f : [a, b] \to [c, d]$ **is a surjection**. Choose any $k \in [c, d]$. Thus $c \leq k \leq d$, or in other words, $f(a) \leq k \leq f(b)$. Therefore, by the Intermediate Value Theorem there exists $\ell \in [a, b]$ for which $f(\ell) = k$. Hence f maps onto k. True for all $k \in [c, d]$ means f maps onto $[c, d]$, i.e. f is a surjection.

• $f : [a,b] \to [c,d]$ **is an injection.** Assume **not**, so there exists **two** $\ell \neq \ell' \in [a,b]$ such that $f(\ell) = f(\ell')$. Without loss of generality, $\ell < \ell'$. Since f is *strictly* increasing we have $f(\ell) < f(\ell')$ a contradiction.

• Hence $f : [a,b] \to [c,d]$ is a bijection and so there exists an inverse $g : [c,d] \to [a,b]$ satisfying $f(g(k)) = k$ for all $k \in [c,d]$ and $g(f(\ell)) = \ell$ for all $\ell \in [a,b]$.

The inverse function g strictly increasing on $[c,d]$ The inverse g is strictly increasing if, and only if,

$$\forall k_1, k_2 \in [c,d], k_1 < k_2 \implies g(k_1) < g(k_2).$$

Assume **not**, so the negation is true, namely

$$\exists\, k_1, k_2 \in [c,d], k_1 < k_2 \quad \text{and} \quad g(k_1) \geq g(k_2).$$

Since f is increasing $g(k_1) \geq g(k_2)$, switched around as $g(k_2) \leq g(k_1)$, implies $f(g(k_2)) \leq f(g(k_1))$ which can be switched to $f(g(k_1)) \geq f(g(k_2))$. But f and g are inverses, so we get $k_1 \geq k_2$ which contradicts $k_1 < k_2$.

Hence assumption false, and so g is strictly increasing.

The inverse function g is continuous on $[c,d]$. *Proof not given 2019-20.*

We need to show that

$$\forall k \in [c,d], \forall \varepsilon > 0, \exists\, \delta > 0 : \forall y,\ |y-k| < \delta \implies |g(y) - g(k)| < \varepsilon.$$

We prove this here only for the open interval (c,d), I leave the question of end points to the interested Student.

Let $k \in (c,d)$, an interior point, be given. Let $\varepsilon > 0$ be given. Choose

$$\delta = \min\left(k - f(g(k) - \varepsilon),\ f(g(k) + \varepsilon) - k\right).$$

Is $\delta > 0$? Start with the trivially true inequalities $g(k) - \varepsilon < g(k) < g(k) + \varepsilon$. Applying f throughout, a *strictly* increasing function, gives

$$\begin{aligned} & f(g(k) - \varepsilon) < f(g(k)) < f(g(k) + \varepsilon), \\ \text{i.e.} \qquad & f(g(k) - \varepsilon) < k < f(g(k) + \varepsilon), \end{aligned}$$

since $f(g(k)) = k$. Rearrange these two inequalities as $k - f(g(k) - \varepsilon) > 0$ and $f(g(k) + \varepsilon) - k > 0$ and we see that $\delta > 0$ as required.

We will use two properties of δ, namely

$$\delta \leq k - f(g(k) - \varepsilon) \quad \text{and} \quad \delta \leq f(g(k) + \varepsilon) - k. \tag{1}$$

We split $|y - k| < \delta$ into the two cases of $y \geq k$ and $y < k$.

Case 1. Assume y satisfies $k \leq y < k + \delta$ then, by the **second** property of δ in (1), we have

$$y < k + (f(g(k) + \varepsilon) - k) = f(g(k) + \varepsilon).$$

i.e. $k \leq y < f(g(k) + \varepsilon)$. Apply g, which we have shown to be a *strictly* increasing function, to get

$$\begin{aligned} g(k) \leq g(y) &< g(f(g(k) + \varepsilon)) \\ &= (g \circ f)(g(k) + \varepsilon) \\ &= g(k) + \varepsilon \end{aligned}$$

using the fact that g is the inverse of f. Hence

$$k \leq y < k + \delta \implies |g(y) - g(k)| < \varepsilon.$$

Case 2 Assume y satisfies $k - \delta < y < k$. Then, by the **first** property of δ in (1),

$$k - (k - f(g(k) - \varepsilon)) < y.$$

i.e. $f(g(k) - \varepsilon) < y < k$. Apply g to get

$$g(f(g(k) - \varepsilon)) < g(y) < g(k).$$

But $g(f(g(k) - \varepsilon)) = (g \circ f)(g(k) - \varepsilon) = g(k) - \varepsilon$ and so

$$g(k) - \varepsilon < g(y) < g(k).$$

Hence

$$k - \delta < y < k \implies |g(y) - g(k)| < \varepsilon.$$

Combine both cases to get $0 < |y - k| < \varepsilon$ implies $|g(y) - g(k)| < \varepsilon$. Hence g is continuous at k.

Since $k \in (c, d)$ is arbitrary, g is continuous on (c, d).

Finally, If f is strictly *decreasing* on $[a, b]$ apply the result above to $\widehat{f}(x) = -f(x)$ a continuous strictly *increasing* function. We then find $\widehat{g}$ the continuous, strictly increasing inverse to $\widehat{f}$. I claim that $g(x) = \widehat{g}(-x)$ is the inverse to f. To see this just check

$$g(f(x)) = \widehat{g}(-f(x)) = \widehat{g}\left(\widehat{f}(x)\right) = x$$

for all $x \in [a, b]$, and

$$f(g(y)) = -\widehat{f}(\widehat{g}(-y)) = -(-y) = y,$$

for all $y \in [c, d]$.

The function $\widehat{g}$ is strictly increasing yet I claim that g is strictly decreasing. To see this let $k_1 < k_2$. Then $-k_2 < -k_1$ and so $\widehat{g}(-k_2) < \widehat{g}(-k_1)$ since $\widehat{g}$ is strictly increasing. Yet by the definition of g we have $g(k_2) < g(k_1)$. So we have shown that $k_1 < k_2 \Rightarrow g(k_1) > g(k_2)$, the definition of g being strictly decreasing. ■

The Inverse Function Theorem should not be a surprise. The *existence* of g will follow from the Intermediate Value Theorem. For every $c < y < d$ there exists, by the IV Thm, $a < x < b : f(x) = y$. Define g by $g(y) = x$. That x is *uniquely defined* is a simple proof by contradiction as is the fact that g is a *strictly increasing function*. The longest part of the proof is to show that g is *continuous*. Yet, recall the 'idea' that a function is continuous if the graph can be drawn without taking the pen off the paper. But this property holds however the graph is described, be it in terms of its distance from the x-axis, i.e. as $f(x)$, or its distance from the y-axis, i.e. as $g(y)$.

I have stated Theorem 2.3.3 for functions on a closed and bounded interval $[a, b]$ because of the use, in the proof, of the IV Thm which requires $[a, b]$. There are yet more versions of Theorem 2.3.3 for intervals such as $[a, b)$, $(a, b]$ or (a, b). And further we can allow $a = -\infty$ or $b = +\infty$. Proofs of these extensions are not given in this course but are used in the following

Example 2.3.4 *Each of the functions* $x^2, x^3, ..., x^n, ...$ *is strictly increasing on* $[0, \infty)$ *and continuous there. Therefore the inverse functions*

$$\sqrt{x}, x^{1/3}, ..., x^{1/n}, ...$$

are well-defined and continuous on $[0, \infty)$*. In fact* $x^{1/n}$ *are continuous on* $\mathbb{R}$ *if* n *is odd.*

Solution is immediate. ■

You will have made use of the logarithm in MATH10242 and many times in other courses but you have had to wait until now for its definition.

Example 2.3.5 *So far we have seen that the function* e^x *is continuous and strictly increasing from* $(-\infty, \infty)$ *to* $(0, \infty)$*. Hence it has an inverse function from* $(0, \infty)$ *to* $(-\infty, \infty)$ *. We denote this function by* $\ln x$*, so* $\ln(e^x) = x$ *for all* $x \in (-\infty, \infty)$ *while* $e^{\ln y} = y$ *for all* $y \in (0, \infty)$*. The Inverse Function Theorem tells us that* $\ln$ *is a continuous strictly increasing function.*

You should ask yourself whether this function, $\ln x$, the *natural logarithm of* x, has the properties you expect. For example, is it true that $\ln a + \ln b = \ln ab$ for all $a, b > 0$?

Having defined $x^{1/n}$ for $x \geq 0, n \geq 1$ we can defined x^r for $x \geq 0$ and $r \in \mathbb{Q}$ by writing $r = m/n$ and setting $x^r = \left(x^{1/n}\right)^m$. By the Product Rule for continuous functions this is continuous. I leave it to the students to check it is strictly increasing.

This leaves the question of defining x^α for $x \geq 0$ and $\alpha \in \mathbb{R}$

A solution could be to take a sequence of rational numbers $\{r_n\}_{n \geq 1}$ converging to α and then define

$$x^\alpha = \lim_{n \to \infty} x^{r_n}.$$

There would be things to prove, i.e. does this limit always exist; is it independent of the sequence $\{r_n\}$ chosen; do the properties of x^{r_n}, i.e. continuous and strictly increasing, translate to x^α?

With the exponential and logarithmic functions we can give an alternative definition of the power of a non-negative number.

Definition 2.3.6 *For* $x \in (0, \infty)$ *and* $\alpha \in \mathbb{R}$ *define* $x^\alpha = e^{\alpha \ln x}$*.*

Then by the Composition Rule for continuous functions $x^\alpha = \exp(\alpha \ln x)$ is continuous in x on $(0, \infty)$ for all real α. I leave it to the students to check that for $\alpha > 0$ it is strictly increasing, while for $\alpha < 0$ it is strictly decreasing.